Option Pricing under Stochastic Volatility, Equity Premium, and Interest Rates in a Complete Market

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Definition (Options)

Options contracts provide the buyer or investor with the right, but not the obligation, to buy or sell an underlying asset at a preset price, called the strike price K, at the expiration time T.

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Definition (Options Pricing)

The process of determining the fair price of an option that helps traders maximize profits and optimize decision-making.

Preliminaries

Definition (Martingale)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let T be a fixed positive number, and let $\mathcal{F}(t), 0 \leq t \leq T$, be a filtration of sub-σ-algebras of $\mathcal{F}$. Consider a stochastic process $M(t)$ adapted to $\mathcal{F}(t)$, each $M(t) \in L^1$, where $0 \leq t \leq T$. If

$$\mathbb{E}[M(t)|\mathcal{F}(s)] = M(s) \text{ for all } 0 \leq s \leq t \leq T$$

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Theorem (Martingale and Risk-neutral measure)

A discounted process is a martingale under risk-neutral measure (a probability measure that assumes all risky assets earn the risk-free rate of return).

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- The BSM equation:

$$V_t = rV - rSV_s - \frac{\sigma^2}{2} S^2 V_{ss}$$

Research Objectives

- **Motivation**: Restrictions of the BSM model
  - Constant variance of stock price
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- **Research Objectives**: build a more robust model that accounts for changes in equity premium and variance of stocks as well as the bond's interest rate.

Assumptions

- No arbitrage allowed: If $P(T) = V(T)$, then $P(t) = V(t) \ \forall \ 0 < t \leq T$
- All processes are pricing processes

Definition (Pricing process)

For any stochastic process $\{V_t\}$ adapted to $\{\mathcal{F}_t\}$, the natural filtration generated by the portfolio process P, then we say V_t is a **pricing process** if there exists a risk-neutral measure Q of the portfolio process P, such that discounted option price process is a martingale.

- No transaction costs: No extra fee when trading.
- Perfect liquidity: Buy or sell any quantity of any asset at any time.

Our Model

We assumed the following system to describe the evolution of the stock price:

$$
\begin{cases}
dS(t) & = (\mu + X(t) + r)S(t)dt + \sqrt{\sigma_s(t)}S(t)dW_1(t) \\
dX(t) & = -\kappa_x X(t)dt + \sigma_x(\rho_x dW_1(t) + \sqrt{1-\rho_x^2}dW_2(t)) \\
d\sigma_s(t) & = \kappa_x(\sigma - \sigma_s(t))dt + \eta\sqrt{\sigma_s(t)}(\rho_s dW_1(t) + \sqrt{1-\rho_s^2}dW_3(t)) \\
dR(t) & = \kappa_R(r - R(t))dt + \xi(\rho_R dW_1(t) + \sqrt{1-\rho_R^2}dW_4(t))
\end{cases}
$$

- $S(t)$: Underlying asset price/stock price
- $X(t)$: Change in equity premium
- $\sigma_S(t)$: Variance of the stock price
- $R(t)$: Bond's interest rate

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- The PDE (4 variables, S, σ_s, X, R):

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\begin{aligned}
V_t = {} & R(V - V_s S - V_x X - V_{\sigma_s}\sigma_s - V_r R) \\
& - \frac{1}{2}(V_{SS}\sigma_s S^2 + V_{\sigma_s\sigma_s}\eta^2\sigma_s + V_{RR}{\sigma_R}^2 + V_{XX}{\sigma_X}^2) \\
& - V_{S\sigma_s}\eta\sigma_s S\rho_S - V_{SR}\sigma_R\sqrt{\sigma_s}S\rho_R - V_{R\sigma_s}\sigma_R\eta\sqrt{\sigma_s}\rho_R\rho_S \\
& - V_{X\sigma_s}\eta\sqrt{\sigma_s}\sigma_X\rho_X\rho_S - V_{XS}\sigma_X\sqrt{\sigma_s}S\rho_X - V_{XR}\sigma_X\sigma_R\rho_R\rho_X
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- Used the finite difference method to approximate the solution of the PDE.

Derivation: Replicating Portfolio Method

Theorem

Hedgeability Theorem states that every derivative is hedgeable if and only if every underlying asset of the derivative is tradeable.

Assuming S, X, σ_s are tradable pricing processes. By Hedgeability theorem, we have every derivative process $V(S, X, \sigma_s, t)$ for some t can be hedged by a portfolio process P given that every pricing process generating the derivative is tradeable, that is,

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$$dP = R(P - \Delta_x X - \Delta_s S - \Delta_{\sigma_s} - \Delta_R R)dt + \Delta_x dX + \Delta_s dS + \Delta_{\sigma_s} d\sigma_s + \Delta_R dR$$

for arbitrary previsible adapted processes $\{\Delta_i\}$ denoting for the trading strategy on each asset i.

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for arbitrary previsible adapted processes $\{\Delta_i\}$ denoting for the trading strategy on each asset i. Hence, setting $V = P$ and given V, P are pricing processes, we have $dV = dP$.
Solve $dV = dP$ by setting Δ terms, we can get the PDE with only deterministic terms left.

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Applying Girsanov's theorem, we are able to find Q.

$$d(DS) = SdD + DdS + dDdS = DS\sqrt{\sigma_s}(\frac{\mu + X}{\sqrt{\sigma_s}}dt + dW_1) =: DS\sqrt{\sigma_s}d\widetilde{W_1}$$

where $\widetilde{W_1}$ denotes for a martingale under new measure Q. And we do the same thing for every other discounted asset process, getting that

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$$\begin{cases} d(DS) = DS\sqrt{\sigma_s}d\widetilde{W_1} \\ d(D\sigma_s) = D\eta\sqrt{\sigma_s}(\rho_s d\widetilde{W_1} + \sqrt{1-\rho_s^2}d\widetilde{W_3}) \\ d(DX) = D\sigma_X(\rho_X d\widetilde{W_1} + \sqrt{1-\rho_X^2}d\widetilde{W_4}) \end{cases}$$

Derivation: Risk-Neutral Measure Method (Cont.)

Note that V is a pricing process, such that its discounted process DV is a martingale, where we notice that

$$d(DV) = DdV + VdD$$

and since DV is a martingale, applying Ito's lemma on DV, the deterministic part of the equation is equal to 0.

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and since DV is a martingale, applying Ito's lemma on DV, the deterministic part of the equation is equal to 0.
Thus, we get the desired PDE

$$V_t = R(V - \sum V_{X_i} X_i) - \frac{1}{2}(V_{SS}\sigma_s S^2 + V_{\sigma_s\sigma_s}\eta^2\sigma_s + V_{RR}{\sigma_R}^2 + V_{XX}{\sigma_X}^2)$$
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Theorem of Hedgeability

Theorem

Suppose a market whose asset processes satisfy No Arbitrage, Frictionless, Free Trading Position and full liquidity of assets. Consider the market consists some value processes $X_i(t)$ *and a riskless interest rate process* R_t *for* $i \in I = \{1, \cdots, n\}, t \in \mathbb{R}^+$. *Denote* V *for arbitrary derivative processes of* $(X_1, \cdots, X_n, R, t)$ *w.r.t.* Q, *a risk-neutral measure. We use* P *to denote portfolio process.*

For any such V, *there exists a portfolio process* $dV = dP$ *if and only if* P *can be written into the form*

$$dP = R(P - \sum_{i \in I} \Delta_i X_i - \Delta_R R)dt + \sum_{i \in I} \Delta_i dX_i + \Delta_R dR \tag{1}$$

Hedgeability Theorem: Proof

$(\implies)$
For the forward proof it suffices to prove that

$$dP = dV \implies dP = R(P - \sum_{i \in I} \Delta_i X_i - \Delta_R R)dt + \sum_{i \in I} \Delta_i dX_i + \Delta_R dR \quad (2)$$

is true. Suppose some pricing processes X_j are not tradeable and unadapted, then it leads to a contradiction to $dV = dP$. Otherwise it must can be written in dP form to be tradeable.

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is true. Suppose some pricing processes X_j are not tradeable and unadapted, then it leads to a contradiction to $dV = dP$. Otherwise it must can be written in dP form to be tradeable.
($\Longleftarrow$)
Consider interest-discounted process $\{X_i', P', V'\}$ for each X_i, P, and V.
Since V' is a pricing process for all V.
We have that $\{X_i', P', V'\}$ are martingales under Q. Therefore, We may apply martingale transformation theorem on V', finding out it could be represented by some discounted portfolio processes P'.

Derivative Estimation

- Derivative estimation for time: $\frac{\partial U}{\partial t} \approx \frac{U_{i,j,m,n}^{t+1} - U_{i,j,m,n}^{t}}{\Delta t}$

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- Second-order single-variable spatial derivative estimation: $\frac{\partial^2 U}{\partial S^2} \approx \frac{U_{i+1,j,m,n}^{t} - 2U_{i,j,m,n}^{t} + U_{i-1,j,m,n}^{t}}{\Delta S^2}$ $\rightarrow$ similar for σ_s, X, and R.

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- Second-order mixed-variable spatial derivative estimation:
  $\frac{\partial^2 U}{\partial S \partial \sigma_s} \approx \frac{U_{j+1,m+1,n}^{t} - U_{j-1,m-1,n}^{t} - U_{j+1,m-1,n}^{t} + U_{j-1,m+1,n}^{t}}{4\Delta S \Delta \sigma_s}$
  $\rightarrow$ similar for other mixed-variable spatial derivatives

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Let M be a transformation matrix. Our goal is to find U^{t+1}.

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European Call and Up-and-out Barrier Call Options

Let strike price $K = 5$, barrier $B = 8$. The expiration time is T = 1, stock price $S = [0, 10]$, variance of the stock price $\sigma_s = [0, 1]$, change in equity premium $X = [-1, 1]$, and interest rate $R = [-0.2, 0.2]$.

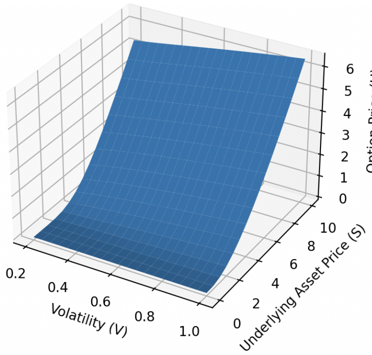

Figure: Call option price when $X = 0.5$ and $R = 0.06$

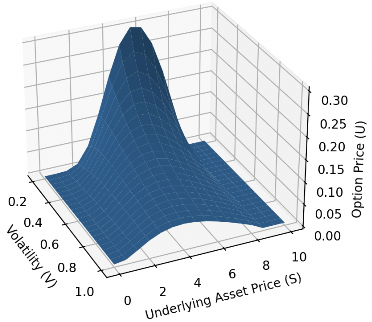

Figure: Barrier option price when $X = 0.5$ and $R = 0.06$

Convergence

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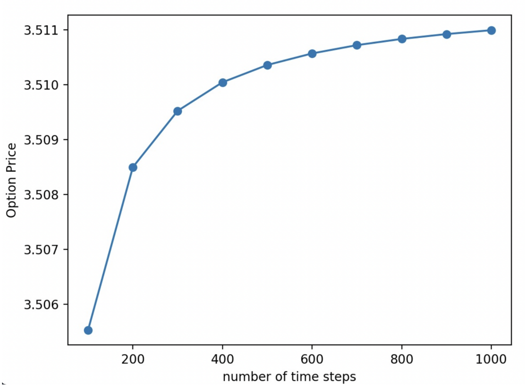
Below is the European call option price at different time steps of Crank-Nicolson scheme when $S = 8.5, \sigma_s = 0.28, X = 0$, and $R = 0.02$.



Future Directions

- Find an accurate numerical approximation to the solution of the Asian option PDE

$$
\begin{aligned}
V_t = {} & R(V - \sum V_{X_i} X_i) - V_A S \\
& - \frac{1}{2}(V_{SS}\sigma_s S^2 + V_{\sigma_s\sigma_s}\eta^2\sigma_s + V_{RR}{\sigma_R}^2 + V_{XX}{\sigma_X}^2) \\
& - V_{S\sigma_s}\eta\sigma_s S\rho_S - V_{SR}\sigma_R\sqrt{\sigma_s}S\rho_R \\
& - V_{R\sigma_s}\sigma_R\eta\sqrt{\sigma_s}\rho_R\rho_S - V_{X\sigma_s}\eta\sqrt{\sigma_s}\sigma_X\rho_X\rho_S \\
& - V_{XS}\sigma_X\sqrt{\sigma_s}S\rho_X - V_{XR}\sigma_X\sigma_R\rho_R\rho_X
\end{aligned}
$$

- Use different numerical schemes and compare results
    - Craig–Sneyd (CS)
    - Hundsdorfer–Verwer (HV)

References

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