OPTION PRICING WITH STOCHASTIC VOLATILITY, EQUITY PREMIUM, AND INTEREST RATE

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Introduction

We introduce a comprehensive model to price options expanding upon the Black-Scholes [1, 5] model, by integrating the Heston model [4] for a time-varying variance on the stock, the Vasicek [6] model for a time-varying interest rate, and the Campbell-Viceira model [2, 3] for a time-varying equity premium.

In particular, we consider the following system of stochastic differential equations (SDEs)

$$\begin{cases} dS(t) = (\mu + X(t) + R(t))S(t)dt + \sqrt{\sigma_s(t)}S(t)dW_1(t) \\ dX(t) = -\kappa_x X(t)dt + \sigma_x(\rho_x dW_1(t) + \sqrt{1-\rho_x^2}dW_2(t)) \\ d\sigma_s(t) = \kappa_s(\sigma - \sigma_s(t))dt + \eta\sqrt{\sigma_s(t)}(\rho_s dW_1(t) + \sqrt{1-\rho_s^2}dW_3(t)) \\ dR(t) = \kappa_r(r - R(t))dt + \sigma_r(\rho_r dW_1(t) + \sqrt{1-\rho_r^2}dW_4(t)). \end{cases} \tag{1}$$

The random variables $S(t), X(t), \sigma_s(t),\ R(t)$ and the parameters $\mu, \sigma_x, \sigma, \eta$, and σ_r represent

- $S(t)$ the underlying asset price, $X(t)$ the deviation in the equity premium from its mean, $\sigma_s(t)$ the volatility, $R(t)$ the risk-free interest rate
- μ the long-term average equity premium on the stock, σ_x the volatility of the equity premium, σ the long-term average volatility of the stock, η the volatility of the volatility, σ_r the volatility of the interest rate
- W_1, W_2, W_3, and W_4 are independent Brownian motions on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ adapted to a filtration $\mathcal{F}_t$
- ρ_x, ρ_s, and ρ_r are the correlation between the stock price and the change in equity premium, between the stock price and the volatility, and between the stock price and the interest rate.
- The parameters κ_x, κ_s, and κ_r correspond to the pressure for the equity premium, volatility, and interest rate to return to their long-term average.

Methodology

Our model assumes an incomplete market since we assume only the stock and the risk-free asset are tradeable. This issue poses challenges when deriving a PDE and formulating an initial value boundary problem. We resolve the issue by treating all pricing processes, including σ_s, X, and R, as tradeable assets, effectively completing the market. To showcase the practical utility of our formulated model, we estimate solutions to our PDEs by imposing terminal and boundary conditions for both European call and knock-out barrier options. We implement three numerical methods: forward Euler, backward Euler, and the Crank-Nicolson schemes.

Main Results

Let V represent the price of an option on a stock S, modeled by system 1. The price of a European derivative on this stock must satisfy the PDE:

$$V_t = R(V - SV_s - XV_x - \sigma_s V_{\sigma_s} - RV_r) - \frac{1}{2}\sigma_s S^2 V_{ss} - \frac{1}{2}\sigma_x^2 V_{xx} - \frac{1}{2}\eta^2\sigma_s V_{\sigma_s\sigma_s} - \frac{1}{2}\sigma_r^2 V_{rr} - \rho_x\sigma_x\sqrt{\sigma_s}SV_{sx}$$
$$- \rho_s\eta\sigma_s SV_{s\sigma_s} - \rho_r\sigma_r\sqrt{\sigma_s}SV_{sr} - \rho_x\rho_s\sigma_x\eta\sqrt{\sigma_s}V_{x\sigma_s} - \rho_x\rho_r\sigma_x\sigma_r V_{xr} - \rho_s\rho_r\eta\sigma_r\sqrt{\sigma_s}V_{\sigma_s r},$$

This result is derived using 2 techniques: replicating portfolio and change of measure.

Replicating Portfolio: Assuming every derivative process V is replicable by some portfolio process P. We take value dynamics of P defined by dP, constructed to replicate the value of the derivative V, with its dynamics given by dV, using Ito's lemma. Equating dP and dV and considering the terms involving dW_1, dW_2, dW_3, and dW_4, we observe equating the portfolio weights (trading strategy) $\Delta_s, \Delta_{\sigma_s}, \Delta_x$, and Δ_r to the partial derivatives V w.r.t. S, σ_S, X, R. Cancelling the terms in the equation $dV = dP$, we arrive at the above PDE.

Risk-Neutral: We start by defining the discount process as $D(t) = e^{-\int R(t)dt}$, which leads to $dD(t) = -R(t)D(t)dt$. Through risk-neutral method we wish to define a probability measure such that the discounted process of every asset is a martingale.
Applying Ito's lemma to the discounted stock price process DS, we derive:

$$d(DS) = DS(\mu + X)dt + DS\sqrt{\sigma_s}dW_1 = DS\sqrt{\sigma_s}\left(\frac{(\mu + X)}{\sqrt{\sigma_s}}dt + dW_1\right) = DS\sqrt{\sigma_s}d\widetilde{W}_1$$

We then define adjusted Brownian motions $\widetilde{W}_2, \widetilde{W}_3, \widetilde{W}_4$ with specific drift terms. By applying Girsanov's theorem, we switch to a new probability measure $\widetilde{\mathbb{P}}$, under which $\widetilde{W}_1, \widetilde{W}_2, \widetilde{W}_3, \widetilde{W}_4$ behave as Brownian motions. Then, $d(DS) = DS\sqrt{\sigma_s}d\widetilde{W}_1$, making the discounted stock process DS a martingale under $\widetilde{\mathbb{P}}$. Under our chosen framework, the discounted processes of equity premium price DX, variance price $D\sigma_s$, and interest rate DR are also martingales under $\widetilde{\mathbb{P}}$. Finally, we express the dynamics of S, X, σ_s, R under the new measure $\widetilde{\mathbb{P}}$, and applying Ito's lemma to DV while setting the dt term to 0, we obtain a PDE that describes our system.

Numerical Results

European Call Options: We used MATLAB to estimate solutions to specific Initial-Value-Boundary Problems (IVBP) by imposing different initial and boundary conditions. Let $K = 5, \rho_s = 0.18, \rho_x = 0.23, \rho_r = 0.21, \eta = 0.027, \sigma_x = 0.011$, and $\sigma_r = 0.019$. The graphs resulting from the given parameters and the Crank-Nicolson method are displayed below.

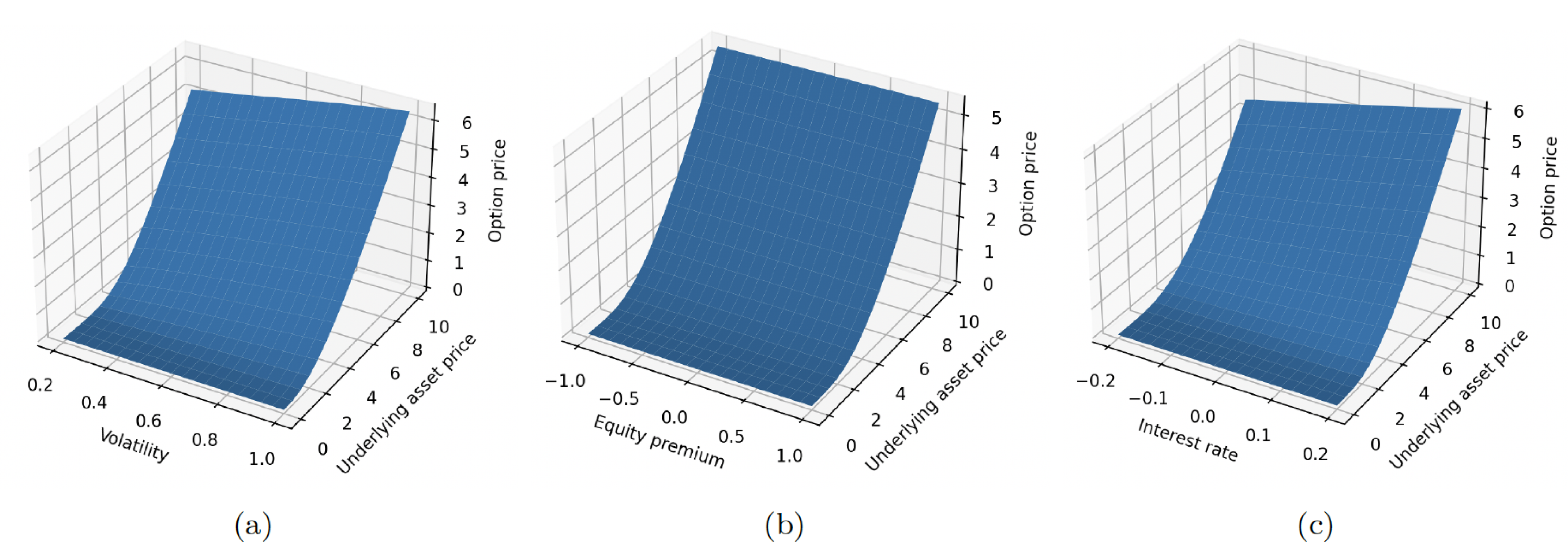

Fig 1. European option price plot of (a) $S\sigma_s$ slice when $X = 0.5$ and $R = 0.04$, (b) SX slice when $\sigma_s = 0.36$ and $R = 0.4$, (c) SR slice when $\sigma_s = 0.36$ and $X = 0.5$

Numerical Results(Cont'd)

European Up-and-out Call Option: We also attempted to estimate the solutions to different options, such as European Up-and-Out Call Options. The figure below shows our estimated solutions. The following is the results:

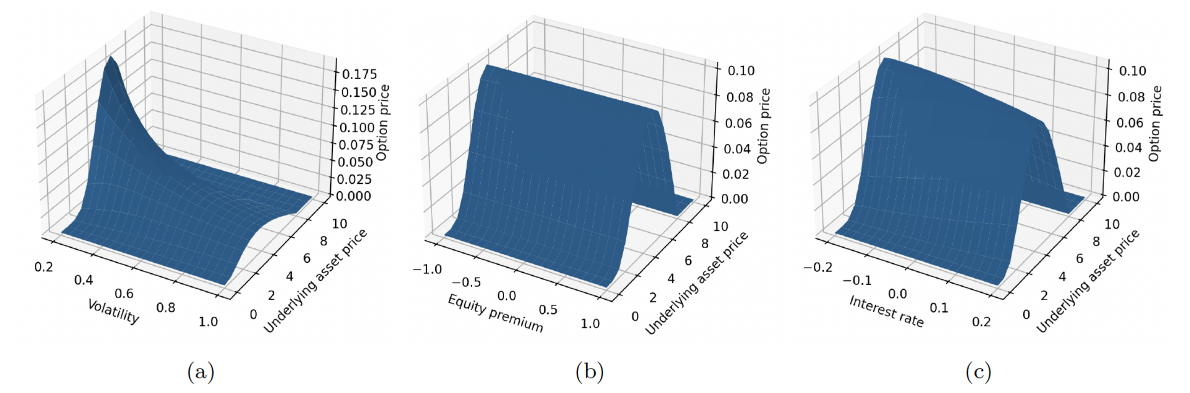

Fig 2. European up-and-out call option price plot of (a) $S\sigma_s$ slice when $X = 0.5$ and $R = 0.04$, (b) SX slice when $\sigma_s = 0.36$ and $R = 0.4$, (c) SR slice when $\sigma_s = 0.36$ and $X = 0.5$

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